A note on group actions on subfactors

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Abstract

We construct approximately inner actions of discrete amenable groups on strongly amenable subfactors of type II_1 with given invariants, and obtain classification results under some conditions. We also study the lifting of the relative χ group.

1 Introduction

In the theory of subfactors initiated by V. F. R. Jones in [14], the analysis of automorphisms and group actions on subfactors has been done by many people. We refer to [2], [7], [9], [10], [11], [16], [17], [18], [19], [21], [22], [23], [24], [30], [34]. Also see [8, Chapter 15].

In [25], we classified approximately inner actions of discrete amenable groups on strongly amenable subfactors of type II_1 by the characteristic invariant and the ν invariant under some assumptions. Among these assumptions, the most important one is the triviality of the algebraic κ invariant. When the algebraic κ invariant is trivial, we can classify approximately inner actions completely. Hence we have to investigate the case when the algebraic κ invariant is not trivial. In this case, we do not know whether there exist actions with given invariants or not. Hence what we should do first is to find a systematic way to construct actions with given invariants. Note that if the algebraic κ invariant is trivial, our characteristic invariant is exactly same as original one in [13], but if the algebraic κ invariant is not trivial, our characteristic invariant may be different from the usual one, and this makes classification more difficult.

In this paper, we construct actions of discrete amenable groups on strongly amenable subfactors of type II_1 with given invariants, and classify actions under an extra assumption on the ν invariant. (We emphasis that we never assume the triviality of the algebraic κ invariant.) The most essential assumption in our theory is that extensibility of the ν invariant to a homomorphism from a whole group. This assumption is similar to that of [20, Theorem 20]. In [20], Kawahigashi, Sutherland and Takesaki have classified the actions of a discrete abelian group G on the injective type III_1 factor. The modular invariant ν appears as the cocycle conjugacy invariant, and this is a homomorphism from a subgroup of G to $\mathbf{R}$. Essential fact in their proof is that ν can be extended to a homomorphism of G due to the divisibility of $\mathbf{R}$. (Originally this idea was due to Connes. See [3, pp.466].)

In subfactor case, we can not expect such property for the ν invariant generally. But if we assume the extensibility of the ν invariant, our proof goes well as in the proof of

[20, Theorem 20]. We remark that our results can be viewed as the generalization of [19, Theorem 4.1].

In appendix, we discuss the lifting of $\chi_a(M,N)$ since we fix one lifting of $\chi_a(M,N)$ to define characteristic invariant.

Acknowledgements. The author is grateful to Professor M. Izumi for informing about [29, Proposition 3.2], Professor Y. Kawahigashi for various comments on this paper. The author is supported by Grant-in-Aid for Scientific Research, Ministry of Education (Japan).

2 Main results

We use notations in [25] freely.

First we recall the definition of cocycle conjugacy invariants for actions considered in [25]. Let $N \subset M$ be a subfactor of type II_1 with finite index, G a discrete group, and α an action of G on $N \subset M$. Throughout this paper, we always make the following assumptions on $N \subset M$.

(A1) $N \subset M$ is extremal,
(A2) $N \subset M$ and $M \subset M_1$ have the trivial normalizer,
(A3) $\mathrm{Ker}\,\Phi = \mathrm{Aut}\,(M,N)$,
(A4) $\chi_a(M,N)$ is a finite group,
(A5) there exists a lifting σ of $\chi_a(M,N)$ to $\mathrm{Aut}\,(M,N)$.

By (A3), every action has trivial Loi invariant. Note that we have many classes of subfactors satisfying the above assumptions, e.g, Jones subfactor with principal graph A_{2n+1} in [14], or subfactors coming from Hecke algebras in [36]. (Also see [17] and [7].)

By [25, Theorem 3.1], we have a Connes-Radon-Nikodym type cocycle $u_{\alpha,\sigma} \in U(N)$ for every $\alpha \in \mathrm{Ker}\,\Phi$ and $\sigma \in \mathrm{Cnt_r}\,(M,N)$. The algebraic κ invariant κ_a is defined by $\kappa_a(h,k) = u^*_{\sigma_k,\sigma_h}$ for $h,k \in \chi_a(M,N)$. We can easily verify that κ_a is a bicharacter of $\chi_a(M,N)$.

For an action α of G, we get cocycle conjugacy invariants in the following way. The first invariant is a normal subgroup $H_\alpha \subset G$, which is a non-strongly outer part of α. Then we get a G-equivariant homomorphism ν_α from H_α to $\chi_a(M,N)$ by $\nu_\alpha(h) = [\alpha_h]$. This ν_α is the second cocycle conjugacy invariant, and we call this the ν invariant. By (A5) α_h has the form $\alpha_h = \mathrm{Ad}\, v_h \sigma_{\nu_\alpha(h)}$ for some unitary $v_h \in U(N)$. Then we get a characteristic invariant $\Lambda(\alpha) = [\lambda_\alpha, \mu_\alpha] \in \Lambda(G, H_\alpha|\kappa_a)$ by the following equations for $g \in G$, $h,k \in H_\alpha$.

$$\alpha_g(v_{g^{-1}hg})u_{\alpha_g,\sigma_{\nu_\alpha(h)}} = \lambda_\alpha(g,h)v_h, \quad v_h\sigma_{\nu_\alpha(h)}(v_k) = \mu_\alpha(h,k)v_{hk}.$$

The pair $\lambda(g,h)$ and $\mu(h,k)$ satisfy the following relations for $h,k,l \in H$ and $g,g_1,g_2 \in G$.

(1) $\mu(h,k)\mu(hk,l) = \mu(k,l)\mu(h,kl)$,
(2) $\lambda(g_1g_2,h) = \lambda(g_1,h)\lambda(g_2,g_1^{-1}hg_1)$,
(3) $\lambda(g,hk)\overline{\lambda(g,h)\lambda(g,l)} = \mu(h,k)\overline{\mu(g^{-1}hg,g^{-1}kg)}$
(4) $\lambda(h,k) = \mu(h,h^{-1}kh)\overline{\mu(k,h)\kappa_a(\nu(k),\nu(h))}$,
(5) $\lambda(e,h) = \lambda(g,e) = \mu(e,k) = \mu(h,e) = 1$.

Equation (4) shows the difference between the usual characteristic invariant and our characteristic invariant. This definition of λ and μ depends on the choice of v_h. To get

rid of this dependence we have to define suitable equivalence relation for (λ, μ). On this point see [25].

Conversely for a given normal subgroup $H \subset G$, $[\lambda, \mu] \in \Lambda(G, H|\kappa_a)$ and $\nu \in \mathrm{Hom}_G(H, \chi_a(M, N))$, we will construct an action α with $H_\alpha = H$, $\Lambda(\alpha) = [\lambda, \mu]$ and $\nu_\alpha = \nu$ in the following proposition.

Proposition 2.1 *Let $N \subset M$ be a strongly amenable subfactor of type II_1, G a discrete amenable group. Assume that ν can be extended to the homomorphism from G. Then for every $[\lambda, \mu] \in \Lambda(G, H|\kappa_a)$ and ν, there exists an action α of G with $H_\alpha = H$, $\Lambda(\alpha) = [\lambda, \mu]$ and $\nu_\alpha = \nu$.*

Proof. By assumptions, we have an extension of ν from G to $\chi_a(M, N)$, which we denote ν again. Hence $g \to \sigma_{\nu(g)}$ is an action of G on $N \subset M$. Let κ_a be the algebraic κ invariant for $N \subset M$, and set $\lambda'(g, n) := \kappa_a(\nu(n), \nu(g))\lambda(g, n)$. Then it is easy to verify that $[\lambda', \mu]$ is in $\Lambda(G, H)$, that is, $[\lambda', \mu]$ is a usual characteristic invariant. Let m be an action of G on the injective type II_1 factor R_0 with the characteristic invariant $[\lambda', \mu]$. Define an action α of G by $\alpha_g := \sigma_{\nu(g)} \otimes m_g$. Then this α is a desired one. $\Box$

On classification of actions, we have the following result.

Theorem 2.2 *Let $N \subset M$, G be as in the previous proposition. Let α and β be approximately inner actions of G. Assume ν_α can be extended to a homomorphism from G. Then α and β are stably conjugate if $H_\alpha = H_\beta$, $\Lambda(\alpha) = \Lambda(\beta)$ and $\nu_\alpha = \nu_\beta$ hold.*

Proof. Set $K := \chi_a(M, N)$. Let $\widetilde{\alpha}$ be an extension of α on $N \rtimes_\sigma K \subset M \rtimes_\sigma K$ defined in [25], and $\widetilde{\widetilde{\alpha}}$ be a natural extension of $\widetilde{\alpha}$ on $\widetilde{\widetilde{N}} \subset \widetilde{\widetilde{M}} := N \rtimes_\sigma K \rtimes_{\widehat{\sigma}} \widehat{K} \subset M \rtimes_\sigma K \rtimes_{\widehat{\sigma}} \widehat{K}$. Let w_k be an implementing unitary of σ in $M \rtimes_\sigma K$, and v_p be an implementing unitary of $\widehat{\sigma}$ in $M \rtimes_\sigma K \rtimes_{\widehat{\sigma}} \widehat{K}$. Then by the definition of $\widetilde{\widetilde{\alpha}}$, we have $\widetilde{\widetilde{\alpha}}_g(x) = \alpha_g(x)$, $\widetilde{\widetilde{\alpha}}_g(w_k) = u_{\alpha_g, \sigma_k} w_k$ and $\widetilde{\widetilde{\alpha}}_g(v_p) = v_p$ for $x \in M$, $k \in K$ and $p \in \widehat{K}$. On the other hand, the second dual action $\widehat{\widehat{\sigma}}$ of σ satisfies $\widehat{\widehat{\sigma}}_k = \mathrm{id}$ on $M \rtimes_\sigma K$ and $\widehat{\widehat{\sigma}}_k(v_p) = \overline{\langle k, p \rangle} v_p$ for $p \in \widehat{K}$.

Takesaki duality theorem says that $\widetilde{\widetilde{N}} \subset \widetilde{\widetilde{M}}$ is isomorphic to $N \otimes B(l^2(K)) \subset M \otimes B(l^2(K))$ via an isomorphism Ψ satisfying the following.

$$
\begin{aligned}
&(1) \qquad (\Psi(\pi_{\widehat{\sigma}} \circ \pi_\sigma(a))\xi)(k) = \sigma_k^{-1}(a)\xi(k), \\
&(2) \qquad (\Psi(\pi_{\widehat{\sigma}}(w_l))\xi)(k) = \xi(l^{-1}k), \\
&(3) \qquad (\Psi(v_p)\xi)(k) = \overline{\langle k, p \rangle}\xi(k),
\end{aligned}
$$

where π_σ is an embedding of M into $M \rtimes_\sigma K$, and $\pi_{\widehat{\sigma}}$ is an embedding of $M \rtimes_\sigma K$ into $M \rtimes_\sigma K \rtimes_{\widehat{\sigma}} \widehat{K}$.

Define a unitary $c_g \in N \otimes B(l^2(K))$ by $(c_g\xi)(k) := u^*_{\alpha_g, \sigma_k^{-1}}\xi(k)$. Since c_g commutes with elements in $N' \otimes \mathbf{C}1$, c_g is indeed in $N \otimes B(l^2(K))$. Moreover since we have

$$
\begin{aligned}
(c_g \alpha_g \otimes id(c_h)\xi)(k) &= u^*_{\alpha_g, \sigma_k^{-1}} \alpha_g(u^*_{\alpha_h, \sigma_k^{-1}})\xi(k) \\
&= u^*_{\alpha_{gh}, \sigma_k^{-1}}\xi(k) \\
&= (c_{gh}\xi)(k),
\end{aligned}
$$

c_g is an $\alpha \otimes \mathrm{id}$ cocycle. Then as in the argument in [22, Section 5], it is shown that $\Psi \circ \widetilde{\widetilde{\alpha}}_g \circ \Psi^{-1} = \mathrm{Ad}\, c_g(\alpha_g \otimes \mathrm{id})$ holds.

On the other hand we have $\Psi \circ \widehat{\widehat{\sigma}}_k \circ \Psi^{-1} = \sigma_k \otimes \mathrm{Ad}\,\rho_k^{-1}$, where ρ is a left regular representation of K.

Here we consider the Connes-Radon-Nikodym type cocycle for $\mathrm{Ad}\,c_g\alpha_g \otimes \mathrm{id}$ and $\sigma_k \otimes \mathrm{Ad}\,\rho_k^{-1}$. Take $0 \neq a \in M_n$ with $\sigma_k(x)a = ax$ for every $x \in M$. By [25, Theorem 3.1], $\alpha_g(a) = u_{\alpha_g,\sigma_k} a$ holds. It is obvious that $\sigma_k \otimes \mathrm{Ad}\,\rho_k^{-1}(x)(a \otimes \rho_k^{-1}) = (a \otimes \rho_k^{-1})x$ holds for every $M \otimes B(l^2(K))$. Here we have the following.

$$\begin{aligned}
(\mathrm{Ad}\,c_g(\alpha_g \otimes \mathrm{id})(a \otimes \rho_k^{-1})\xi)(l) &= (c_g(\alpha_g(a) \otimes \rho_k^{-1})c_g^*\xi)(l) \\
&= u^*_{\alpha_g,\sigma_l^{-1}} \alpha_g(a)(c_g^*\xi)(kl) \\
&= u^*_{\alpha_g,\sigma_l^{-1}} u_{\alpha_g,\sigma_k} a u_{\alpha_g,\sigma_{kl}^{-1}} \xi(kl) \\
&= u^*_{\alpha_g,\sigma_l^{-1}} u_{\alpha_g,\sigma_k} \sigma_k(u_{\alpha_g,\sigma_{kl}^{-1}}) a\xi(kl) \\
&= u^*_{\alpha_g,\sigma_l^{-1}} u_{\alpha_g,\sigma_l^{-1}} a\xi(kl) \\
&= (a \otimes \rho_k^{-1}\xi)(l).
\end{aligned}$$

By [25, Theorem 3.1], the above equality implies $u_{\mathrm{Ad}\,c_g(\alpha\otimes\mathrm{id}),\sigma_k\otimes\mathrm{Ad}\,\rho_k^{-1}} = 1$ holds for every $g \in G$ and $k \in K$. Hence by replacing α and β if necessary, we may assume that $u_{\alpha_g,\sigma_k} = 1$ and $u_{\beta_g,\sigma_k} = 1$ hold for every $g \in G$ and $k \in K$. This especially implies $\alpha_g\sigma_k = \sigma_k\alpha_g$ and $\beta_g\sigma_k = \sigma_k\beta_g$.

Define two new actions $\bar{\alpha}$ and $\bar{\beta}$ by $\bar{\alpha}_g := \alpha_g\sigma_{\nu(g)}^{-1}$ and $\bar{\beta}_g := \beta_g\sigma_{\nu(g)}^{-1}$. Since α and β commute with σ, $\bar{\alpha}$ and $\bar{\beta}$ are indeed actions of G. By construction of $\bar{\alpha}$ and $\bar{\beta}$, it is easy to see $H_\alpha = \bar{\alpha}^{-1}(\mathrm{Cnt}\,(M,N)) = \bar{\alpha}^{-1}(\mathrm{Int}\,(M,N)) = \bar{\beta}^{-1}(\mathrm{Cnt}\,(M,N)) = \bar{\beta}^{-1}(\mathrm{Int}\,(M,N))$.

Next we compute $\Lambda(\bar{\alpha})$. Take $m \in H$ and $v_m \in U(N)$ with $\alpha_m = \mathrm{Ad}\,v_m\sigma_{\nu(m)}$. In this case, we have $\mathrm{Ad}\,\bar{\alpha}_m = \mathrm{Ad}\,v_m$ for $m \in H$. Moreover since

$$\begin{aligned}
1 &= u_{\alpha_h,\sigma_k} \\
&= u_{\mathrm{Ad}\,v_m\sigma_{\nu(m)},\sigma_k} \\
&= \mathrm{Ad}\,v_m(u_{\sigma_{\nu(m)},\sigma_k})u_{\mathrm{Ad}\,v_m,\sigma_k} \\
&= \overline{\kappa_a(k,\nu(m))}v_m\sigma_k(v_m^*)
\end{aligned}$$

holds, we have $\sigma_k(v_m) = \overline{\kappa_a(k,\nu(m))}v_m$.

First we compute $\lambda_{\bar{\alpha}}$. Since we have $u_{\alpha_g,\sigma_k} = 1$, $\alpha_g(v_{g^{-1}ng}) = \lambda_\alpha(g,n)v_n$ holds by the definition of λ_α. Then we get

$$\begin{aligned}
\bar{\alpha}_g(v_{g^{-1}ng}) &= \alpha_g\sigma_{\nu(g)}^{-1}(v_{g^{-1}ng}) \\
&= \overline{\kappa_a(\nu(g)^{-1},\nu(n))}\alpha_g(v_{g^{-1}ng}) \\
&= \kappa_a(\nu(g),\nu(n))\lambda_\alpha(g,n)v_n,
\end{aligned}$$

and $\lambda_{\bar{\alpha}}(g,n) = \kappa_a(\nu(g),\nu(n))\lambda_\alpha(g,n)$ holds. Next we compute $\mu_{\bar{\alpha}}$. By the definition of μ_α, we have $v_m\sigma_{\nu(m)}(v_n) = \mu_\alpha(m,n)v_{mn}$, $m,n \in H$. Hence we get $v_mv_n = \kappa_a(\nu(m),\nu(n))\mu_\alpha(m,n)v_{mn}$ and consequently $\mu_{\bar{\alpha}}(m,n) = \kappa_a(\nu(m),\nu(n))\mu_\alpha(m,n)$.

Similar computation holds for $\bar{\beta}$, and by the assumption $\Lambda(\alpha) = \Lambda(\beta)$, we get $\Lambda(\bar{\alpha}) = \Lambda(\bar{\beta})$. Hence $\bar{\alpha}$ and $\bar{\beta}$ are cocycle conjugate by [25, Theorem 5.1]. Then $\bar{\alpha}$ and $\bar{\beta}$ are stably conjugate, hence there exists an automorphism $\theta \in \mathrm{Aut}\,(M \otimes B(l^2(G)), N \otimes B(l^2(G)))$ with $\theta \circ (\bar{\beta}_g \otimes \mathrm{Ad}\,\varrho_g) \circ \theta^{-1} = \bar{\alpha}_g$, where ϱ is a right regular representation of G.

To prove the main theorem, we need the following proposition. In the following proposition, $N \subset M$ can be an arbitrary subfactor of finite index.

Proposition 2.3 *Let σ be a non-strongly outer automorphism. Take $0 \neq a \in M_n$ such that $\sigma(x)a = ax$ holds for every $x \in M$. Then $v \in M$ is in M^σ if and only if $va = av$ holds.*

Proof. First assume that $v \in M^\sigma$. Then we have $va = \sigma(v)a = av$. Conversely assume that $va = av$ holds. Then $\sigma(v)a = av = va$ holds. Hence we have $\sigma(v)aa^* = vaa^*$. Here aa^* is in $M' \cap M_n$. Let E be the minimal conditional expectation from M_n onto M. Then we get $\sigma(v)E(aa^*) = E(\sigma(v)aa^*) = E(vaa^*) = vE(aa^*)$, and $E(aa^*) \in M \cap M' = \mathbf{C}$. Since a is not zero, $E(aa^*)$ is a non-zero scalar. Hence v is in M^σ. □

Remark. The above proposition can be regarded as a subfactor-analogue of the characterization of the centralizer of type III factors. Namely let M be a type III factor, ϕ a faithful normal state of M. Then a is in M_ϕ if and only if $[\phi, a] = 0$.

We continue the proof of Theorem 2.2. Since an outer action of a finite group is stable, we can find a unitary $w \in N \otimes B(l^2(G))$ such that $w^*\sigma_k \otimes \mathrm{id}(w) = u_{\theta,\sigma_k\otimes \mathrm{id}}$. Hence $\theta \circ \sigma_k \otimes \mathrm{id} \circ \theta^{-1} = \mathrm{Ad}\, u_{\theta,\sigma_k\otimes\mathrm{id}} \circ \sigma_k \otimes \mathrm{id} = \mathrm{Ad}\, w^* \circ \sigma_k \otimes \mathrm{id} \circ \mathrm{Ad}\, w$ holds. If we can prove that $w\bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g(w^*)$ is in $(M \otimes B(l^2(G)))^K$, then $w\bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g(w^*)$ is an $\alpha \otimes \mathrm{Ad}\, \varrho = \bar{\alpha}\sigma \otimes \mathrm{Ad}\, \varrho$ cocycle and

$$\begin{aligned}
\bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g \circ \theta \circ \sigma_{\nu(g)} \otimes \mathrm{id} \circ \theta^{-1} &= \bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g \circ \mathrm{Ad}\, w^* \circ \sigma_{\nu(g)} \otimes \mathrm{id} \circ \mathrm{Ad}\, w \\
&= \mathrm{Ad}\, (\bar{\alpha} \otimes \mathrm{Ad}\, \varrho_g(w^*))\bar{\alpha}_g\sigma_{\nu(g)} \otimes \mathrm{Ad}\, \varrho_g \circ \mathrm{Ad}\, w \\
&= \mathrm{Ad}\, w^* \circ \mathrm{Ad}\, (w\bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g(w^*))\alpha_g \otimes \mathrm{Ad}\, \varrho_g \circ \mathrm{Ad}\, w
\end{aligned}$$

holds and we have the following.

$$\begin{aligned}
\alpha \otimes \mathrm{Ad}\, \varrho &= \bar{\alpha} \otimes \mathrm{Ad}\, \varrho \circ \sigma \otimes \mathrm{id} \\
&\sim \bar{\alpha} \otimes \mathrm{Ad}\, \varrho \circ \theta \circ \sigma \otimes \mathrm{id} \circ \theta^{-1} \\
&= \theta \circ \bar{\beta} \otimes \mathrm{Ad}\, \varrho \circ \theta^{-1}\theta \circ \sigma \otimes \mathrm{id} \circ \theta^{-1} \\
&= \theta \circ \bar{\beta}\sigma \otimes \mathrm{Ad}\, \varrho \circ \theta \\
&= \theta \circ \beta \otimes \mathrm{Ad}\, \varrho \circ \theta^{-1}.
\end{aligned}$$

Hence α and β are stably conjugate. So we only have to prove that $w\bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g(w^*)$ is in $(M \otimes B(l^2(G)))^K$.

It is easy to see $u_{\alpha_g\otimes \mathrm{Ad}\, \varrho_g,\sigma_k\otimes\mathrm{id}} = 1$, hence $u_{\bar{\alpha}_g\otimes\mathrm{Ad}\,\varrho_g,\sigma_k\otimes\mathrm{id}} = u_{\alpha_g\sigma^{-1}_{\nu(g)}\otimes\mathrm{Ad}\,\varrho_g,\sigma_k\otimes\mathrm{id}} = \kappa_a(k, \nu(g))$ holds. In the same way, we can see $u_{\bar{\beta}_g\otimes\mathrm{Ad}\,\varrho_g,\sigma_k\otimes\mathrm{id}} = \kappa_a(k, \nu(g))$. Hence

$$\begin{aligned}
u_{\bar{\alpha}_g\otimes\mathrm{Ad}\,\varrho_g,\mathrm{Ad}\, w^*\circ\sigma_k\otimes\mathrm{id}\circ\mathrm{Ad}\, w} &= u_{\theta\circ\bar{\beta}_g\otimes\mathrm{Ad}\,\varrho_g\circ\theta^{-1},\theta\circ\sigma_k\otimes\mathrm{id}\circ\theta^{-1}} \\
&= \theta(u_{\bar{\beta}_g\otimes\mathrm{Ad}\,\varrho_g,\sigma_k\otimes\mathrm{id}}) \\
&= \kappa_a(k, \nu(g))
\end{aligned}$$

holds. Take $0 \neq a \in M_n \otimes B(l^2(G))$ such that $\sigma_k \otimes \mathrm{id}(x)a = ax$ holds for every $x \in M \otimes B(l^2(G))$. Then $\bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g(a) = \kappa_a(k, \nu(g))a$ holds by [25, Theorem 3.1]. Since $\mathrm{Ad}\, w^* \circ \sigma_k \otimes \mathrm{id} \circ \mathrm{Ad}\, w(x)w^*aw = w^*awx$, we also have $\bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g(w^*aw) = \kappa_a(k, \nu(g))w^*aw$. From these two equalities, we get $\bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g(w^*)a\bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g(w) = w^*aw$. Hence $w\bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g(w^*)$ satisfies the condition in Proposition 2.3, $w\bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g(w^*)$ is in a fixed point algebra $(M \otimes B(l^2(G)))^K$. □

Corollary 2.4 *If G is a finite group in Theorem 2.2, then α and β are cocycle conjugate if and only if $H_\alpha = H_\beta$, $\Lambda(\alpha) = \Lambda(\beta)$ and $\nu_\alpha = \nu_\beta$ hold.*

Proof. By Theorem 2.2, $\alpha \otimes \mathrm{Ad}\,\varrho$ and $\beta \otimes \mathrm{Ad}\,\varrho$ are conjugate. But in the same way as in the proof of [15, Lemma 6.5], we can prove that α is cocycle conjugate to $\alpha \otimes m$, where m is an outer action of G/H on the injective type II_1 factor R_0 and we regard m as an action of G in the natural way. Hence α and β are cocycle conjugate since m and $m \otimes \mathrm{Ad}\,\varrho$ are cocycle conjugate. $\square$

In the rest of this paper, we treat examples which satisfy the assumption in Theorem 2.2. The first example is taken from [19, Theorem 4.1].

Example 2.5 We consider the case $G = \mathbf{Z}$. Take $\alpha \in \mathrm{Aut}\,(M, N)$. Let p be a strongly outer period of α. Set $\sigma := \alpha^p$. Then ν_α is given by $\nu_\alpha(pm) = [\sigma^m]$. Let n be an outer period of σ. Here assume $(p, n) = 1$. Then we can find $k, l \in \mathbf{Z}$ such that $pk + nl = 1$. Set $\nu(g) := [\sigma^{gk}]$. Then we have $\nu(p) = [\sigma^{pk}] = [\sigma^{-nl+1}] = [\sigma]$, hence ν_α can be extended to a homomorphism from $\mathbf{Z}$.

Example 2.6 Assume that G is of the form $G = H_\alpha \rtimes K$. For $(h, k) \in G = H_\alpha \rtimes K$, define $\nu(h, k) := \nu_\alpha(h)$. Then by using the fact $\nu_\alpha(knk^{-1}) = \nu_\alpha(n)$, we get

$$\begin{aligned}
\nu((h_1, k_1)(h_2, k_2)) &= \nu(h_1 k_1 h_2 k_1^{-1}, k_1 k_2) \\
&= \nu_\alpha(h_1 k_1 h_2 k_1^{-1}) \\
&= \nu_\alpha(h_1)\nu_\alpha(k_1 h_2 k_1^{-1}) \\
&= \nu(h_1, k_1)\nu(h_2, k_2).
\end{aligned}$$

Hence we can extend ν_α to the homomorphism from G, and we can apply main theorem.

A On liftings of the relative χ group

In [25] and this paper, we fixed a lifting of $\chi_a(M, N)$ to $\mathrm{Aut}\,(M, N)$ for classification of group actions on subfactors. But it seems that there are no way to choose the natural lifting. In this appendix, we show that there exists a natural choice of a lifting by using the algebraic κ invariant.

Take a lifting σ. Then the algebraic κ invariant $\kappa_a(h, k)$ is defined as $\kappa_a(h, k) := u^*_{\sigma_k, \sigma_h}$. To specify σ, we denote this κ_a by $\kappa_a^\sigma(h, k)$.

Fix σ and take another lifting $\tilde{\sigma}$. Then we can find a unitary $u_h \in U(N)$ with $\mathrm{Ad}\,u_h \sigma_h = \tilde{\sigma}_h$. Since $\tilde{\sigma}$ is a lifting, there exists a 2-cocycle $\mu(h, k) \in Z^2(\chi_a(M, N), \mathbf{T})$ with $u_h \sigma_h(u_k) = \mu(h, k) u_{hk}$. Here we compute $\kappa_a^{\tilde{\sigma}}$. Then

$$\begin{aligned}
\overline{\kappa_a^{\tilde{\sigma}}(h, k)} &= u_{\tilde{\sigma}_k, \tilde{\sigma}_h} \\
&= u_{\mathrm{Ad}\,u_k \sigma_k, \mathrm{Ad}\,u_h \sigma_h} \\
&= u_{\mathrm{Ad}\,u_k \sigma_k, \mathrm{Ad}\,u_h} \mathrm{Ad}\,u_h(u_{\mathrm{Ad}\,u_k \sigma_k, \sigma_h}) \\
&= u_k \sigma_k(u_h) u_k^* u_h^* \mathrm{Ad}\,u_h(\mathrm{Ad}\,u_k(u_{\sigma_k, \sigma_h}) u_{\mathrm{Ad}\,u_k, \sigma_h}) \\
&= u_k \sigma_k(u_h) u_k^* u_{\sigma_k, \sigma_h} u_k \sigma_h(u_k^*) u_h^* \\
&= \overline{\kappa_a^\sigma(h, k)\mu(h, k)}\mu(k, h)
\end{aligned}$$

holds, hence we get $\kappa_a^{\tilde{\sigma}}(h,k)=\kappa_a^{\sigma}(h,k)\mu(h,k)\overline{\mu(k,h)}$.

Here assume $\kappa_a^{\tilde{\sigma}}=\kappa_a^{\sigma}$. Then we get $\mu(h,k)=\mu(k,h)$. By [29, Proposition 3.2], μ is a coboundary, so we can choose u_h as a σ-cocycle. Hence we get the following proposition.

Proposition A.1 *Let σ and $\tilde{\sigma}$ be liftings of $\chi_a(M,N)$ to $\mathrm{Aut}\,(M,N)$. If $\kappa_a^{\sigma}=\kappa_a^{\tilde{\sigma}}$ holds, then $\tilde{\sigma}$ is a cocycle perturbation of σ.*

By Proposition A.1, we can find a lifting σ up to cocycle perturbation once we fix the algebraic κ invariant.

In the next proposition, we do not assume $\mathrm{Ker}\,\Phi=\mathrm{Aut}\,(M,N)$. Every $\theta\in\mathrm{Aut}\,(M,N)$ induces an automorphism $\chi_a(\theta)$ of $\chi_a(M,N)$ by $\chi_a(\theta)([\sigma]):=[\theta\circ\sigma\circ\theta^{-1}]$.

Proposition A.2 *Let σ be a lifting of $\chi_a(M,N)$ to $\mathrm{Aut}\,(M,N)$. Assume that $\kappa_a^{\sigma}(h,k)=\kappa_a^{\sigma}(\chi_a(\theta)(h),\chi_a(\theta)(k))$ holds for every $\theta\in\mathrm{Aut}\,(M,N)$. Then there exists a $\sigma_{\chi_a(\theta)(\cdot)}$-cocycle w_h such that $\theta\circ\sigma_h\circ\theta^{-1}=\mathrm{Ad}\,w_h\sigma_{\chi_a^{\sigma}(\theta)(h)}$ holds.*

Proof. Take a unitary w_h with $\mathrm{Ad}\,w_h\sigma_{\chi_a(\theta)(h)}=\theta\circ\sigma_h\circ\theta^{-1}$. Then there exists a 2-cocycle $\mu(h,k)$ satisfying $w_h\sigma_{\chi_a(\theta)(h)}(w_k)=\mu(h,k)w_{hk}$. On one hand, we have $u^*_{\theta\circ\sigma_h\circ\theta^{-1},\theta\circ\sigma_k\circ\theta^{-1}}=\theta(u_{\sigma_h,\sigma_k})^*=\kappa_a^{\sigma}(k,h)$. On the other hand, we have

$$\begin{aligned}u^*_{\theta\circ\sigma_h\circ\theta^{-1},\theta\circ\sigma_k\circ\theta^{-1}} &= u^*_{\mathrm{Ad}\,w_h\sigma_{\chi_a(\theta)(h)},\mathrm{Ad}\,w_k\sigma_{\chi_a(\theta)(k)}}\\ &= \kappa_a^{\sigma}(\chi_a(\theta)(k),\chi_a(\theta)(h))\mu(k,h)\overline{\mu(h,k)}.\end{aligned}$$

By assumption on κ_a^{σ}, we can choose w_h as a cocycle as the same reason in the proof of Proposition A.1. □

The assumption on κ_a in the above proposition is satisfied when (1) κ_a is trivial, (2) $\chi_a(M,N)$ is a cyclic group. The former is obvious. The reason of latter is following. It is easy to see $\kappa_a^{\sigma}(h,h)=\kappa_a^{\sigma}(\chi_a(\theta)(h),\chi_a(\theta)(h))$ holds from the above computation. If $\chi_a(M,N)$ is cyclic and g is a generator of $\chi_a(M,N)$, then

$$\begin{aligned}\kappa_a^{\sigma}(g^m,g^n) &= \kappa_a^{\sigma}(g,g)^{mn}\\ &= \kappa_a^{\sigma}(\chi_a(\theta)(g),\chi_a(\theta)(g))^{mn}\\ &= \kappa_a^{\sigma}(\chi_a(\theta)(g^m),\chi_a(\theta)(g^n))\end{aligned}$$

holds.

References

[1] Chen, J., *The Connes invariant $\chi(M)$ and cohomology of groups*, preprint, (1993).

[2] Choda, M., and Kosaki., H., *Strongly outer actions for an inclusion of factors*, J. Func. Anal. **122** (1994), 315–332.

[3] Connes, A., *On the classification of von Neumann algebras and their automorphisms*, Symposia Mathematica **XX** (1976), 435–478.

[4] Connes, A., *Outer conjugacy classes of automorphisms of factors*, Ann. Sci. Ec. Norm. Sup. **8** (1975), 383–420.

[5] Connes, A., *Sur la classification des facteurs de type II*, C. R. Acad. Sci. Paris **281** (1975), 13–15.

[6] Connes, A., *Periodic automorphisms of the hyperfinite factor of type II_1*, Acta. Sci. Math. **39** (1977), 39–66.

[7] Evans, D. E. and Kawahigashi, Y., *Orbifold subfactors from Hecke algebras,* Comm. Math. Phys., **165** (1994), 445–484.

[8] Evans, D. E. and Kawahigashi, Y., *Quantum symmetries on operator algebras*, Oxford University Press, (1998).

[9] Goto, S., *Orbifold construction for non-AFD subfactors*, Internat. J. Math., **5** (1994), 725–746.

[10] Goto, S., *Symmetric flat connections, triviality of Loi's invariant and orbifold subfactors*, Publ. RIMS **31** (1995), 609–624.

[11] Goto, S., *Commutativity of automorphisms of subfactors modulo inner automorphisms*, Proc. Amer. Math. Soc. **124** (1996), 3391–3398.

[12] Jones, V. F. R., *A factor anti-isomorphic to itself but without involutory anti-automorphisms*, Math. Scan. **46** (1980), 103–117.

[13] Jones, V. F. R., *Actions of finite groups on the hyperfinite type II_1 factor.* Memoirs of Amer. Math. Soc. **237**, (1980).

[14] Jones, V. F. R., *Index for subfactors*, Invent. Math. **72** (1983), 1–25.

[15] Katayama, Y., Sutherland, C., and Takesaki, M., *The characteristic square of a factor and the cocycle conjugacy of discrete group actions on factors*, Invent. Math. **132** (1998), 331–380.

[16] Kawahigashi, Y., *Centrally trivial automorphisms and an analogue of Connes's $\chi(M)$ for subfactors*, Duke Math. **71** (1993), 93–118.

[17] Kawahigashi, Y., *On flatness of Ocneanu's connections on the Dynkin diagrams and classification of subfactors*, J. Func. Anal. **127** (1995), 63–107.

[18] Kawahigashi, Y., *Orbifold subfactors, central sequences and the relative Jones invariant κ*, Inter. Math. Res. Not. 129–140, (1995)

[19] Kawahigashi, Y., *Classification of approximately inner automorphisms of subfactors*, Math. Annal. **308** (1997), 425–438.

[20] Kawahigashi, Y., Sutherland, C., and Takesaki, M., *The structure of the automorphism group of an injective factor and the cocycle conjugacy of discrete abelian group actions*, Acta. Math. **169** (1992), 105–130.

[21] Kosaki, H., *Automorphisms in the irreducible decompositions of sectors*, Quantum and non-commutative analysis, (ed. H. Araki et al.), Kluwer Academic (1993), 305–316.

[22] Kosaki, H., *Sector theory and automorphisms for factor-subfactor pairs*, J. Math. Soc. Japan **48** (1996), 427–454.

[23] Loi, P., *On automorphisms of subfactors*, J. Func. Anal. **141** (1996), 275–293.

[24] Loi, P., *A note on commuting squares arising from automorphisms on subfactors*, Internat. J. Math., **10** (1999), 207–214.

[25] Masuda, T., *Classification of approximately inner actions of discrete amenable groups on strongly amenable subfactors*, preprint (1999).

[26] Ocneanu, A., *Actions of discrete amenable groups on von Neumann algebras*, Lecture Notes in Math. **1138** (1985), Springer, Berlin.

[27] Ocneanu, A., *Quantized group string algebras and Galois theory for algebras*, in "Operator algebras and applications, Vol. 2 (Warwick, 1987)," London Math. Soc. Lect. Note Series Vol. 136, Cambridge University Press, 1988, pp. 119–172.

[28] Ocneanu, A., *Quantum symmetry, differential geometry of finite graphs and classification of subfactors*, University of Tokyo Seminary Notes **45**, (Notes recorded by Y. Kawahigashi.) (1991).

[29] Olsen, D., Pedersen, G. K., and Takesaki, M., *Ergodic actions of compact abelian groups*, Jour. Op. Th. **3** (1980), 237–269.

[30] Popa, S., *Classification of actions of discrete amenable groups on amenable subfactors of type II*, preprint, (1992).

[31] Popa, S., *Classification of amenable subfactors of type II*, Acta. Math. **172** (1994), 163–255.

[32] Sutherland, C. and Takesaki, M., *Actions of discrete amenable groups on injective factors of type* III_λ, $\lambda \neq 1$. Pac. J. Math. **137** (1989), 405–444.

[33] Takesaki, M., *Duality for crossed products and the structure of von Neumann algebras of type III*, Acta. Math. **131** (1973), 249–310.

[34] Xu, F., *Orbifold construction in subfactors,* Comm. Math. Phys. **166** (1994), 237–254.

[35] Yamagami, S., *A note on Ocneanu's approach to Jones index theory*, Internat. J. Math. **4** (1993), 859–871.

[36] Wenzl, H., *Hecke algebras of type* A_n *and subfactors*, Invent. Math. **92** (1988), 349–383.